FINAL: ALGEBRA I

Date: **6th November 2015**
The Total points is **110** and the maximum you can score is **100** points.

(1) (5+5+5+5+5+5=30 points) Prove or disprove
 (a) The groups $(\mathbb{C}^*, \cdot)$ and $(\mathbb{R}^*, \cdot)$ are isomorphic.
 (b) The groups D_6 and S_3 are isomorphic.
 (c) Let G be a group and $H \leq G$. Then $H \leq C_G(H)$ where $C_G(H)$ is the centralizer of H in G.
 (d) Let G and H be groups. Then $Z(G \times H) = Z(G) \times Z(H)$ where $Z(G)$ denotes the center of G.
 (e) Let H and K be subgroups of a group G. Then HK defined to be $\{g \in G : g = hk \text{ for some } h \in H \text{ and } k \in K\}$ is a group.
 (f) Let G be a group, $H \trianglelefteq G$ and $K \trianglelefteq H$ then $K \trianglelefteq G$.

(2) (5+5+10=20 points) Let G be a group and A be a set. What does it mean to say that G acts on A? When is this action called transitive and when is it called faithful? Show that the action of A_n on the set $\{1, 2, \ldots, n\}$ given by $\sigma \cdot i = \sigma(i)$ for $\sigma \in A_n$ and $1 \leq i \leq n$ is both transitive and faithful for $n \geq 3$.

(3) (5+15=20 points) Define a simple group. Show that a group of order 462 is not simple.

(4) (5+15=20 points) Define a solvable group. Let p be a prime and $n \geq 1$. Show that any group of order p^n is solvable.

(5) (5+15=20 points) Define (external) semidirect product. Let H be a group. Show that there exists a group G such that $H \trianglelefteq G$ and given any automorphism σ of H there exist a $g \in G$ such that $\sigma(h) = ghg^{-1}$ for all $h \in H$.